

GENERALIZED *TONNETZE*

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**ABSTRACT.** We study a generalization of the classical Riemannian *Tonnetz* to  $N$ -tone equally tempered scales (for all  $N$ ) and arbitrary triads. We classify all the spaces which result. The torus turns out to be the most common possibility, especially as  $N$  grows. Other spaces include 2-simplices, tetrahedra boundaries, and the harmonic strip (in both its cylinder and Möbius band variants). The final and most exotic space we find is something we call a ‘circle of tetrahedra boundaries’. These are the *Tonnetze* for spaces of triads which contain a tritone. They are closely related to Peck’s Klein bottle *Tonnetz*.

**Keywords:** *Tonnetz*,  $N$ -tone equal temperament, space of triads, torus, simplicial complex, geometry of chords, generalized scales

## 1. INTRODUCTION

Geometric aspects of chord spaces have been considered by many authors, for instance [4, 7, 15, 14, 12, 8, 10, 16]. We continue this tradition by enumerating all possible geometries that arise as *generalized Tonnetze* in the following way. To any fixed, unordered, three-note chord  $\{a, b, c\}$  in  $\mathbb{Z}/N$  one may associate a 2-dimensional simplicial complex. Its 2-simplices are the mod  $N$  transpositions and inversions of  $\{a, b, c\}$ , its 1-simplices are two-element subsets of transpositions and inversions of  $\{a, b, c\}$ , and a 0-simplex is an element of  $\mathbb{Z}/N$ . In Theorem 32, we prove that these simplicial complexes comprise the following list: 2-simplices, tetrahedra boundaries, tori, cylinders, Möbius bands, and circles of tetrahedra boundaries.

A prime example of such a simplicial complex is the neo-Riemannian *Tonnetz*, which is obtained by taking  $\{a, b, c\}$  to be any major or minor triad. The 0-simplices are the elements of  $\mathbb{Z}/12$ , while the 1-simplices are major thirds  $\{x, x+4\}$ , minor thirds  $\{y, y+3\}$  and perfect fifths  $\{z, z+7\}$  for  $x, y, z \in \mathbb{Z}/12$ . The 2-simplices are the major and minor triads. We call this simplicial complex  $C(3, 4, 5)$  because the step intervals of the minor chord are 3, 4, and 5. See Figure 6 for a representation of this torus in the plane (opposite sides are identified to make the torus). We emphasize that the construction of such simplicial complexes is direct, *it does not rely on any planar representation as an intermediate step*.

Indeed, there may not even be a canonical planar representation. For example, if we begin with  $\{0, 1, 3\}$  in the mod 6 universe  $\mathbb{Z}/6$ , then the resulting simplicial complex consists of boundaries of three tetrahedra boundaries

$$\begin{aligned} \{0134\} &= \{\{013\}, \{014\}, \{034\}, \{134\}\} \\ \{1245\} &= \{\{124\}, \{125\}, \{145\}, \{245\}\} \\ \{2350\} &= \{\{235\}, \{230\}, \{250\}, \{350\}\} \end{aligned}$$

glued together along 1-simplices of the form  $\{x, x+3\}$ . We call this simplicial complex  $C(1, 2, 3)$  because the step intervals of  $\{0, 1, 3\}$  in  $\mathbb{Z}/6$ , from smallest to largest, are 1, 2, and 3.

It turns out that  $C(1, 2, 3)$  is closely related to Peck’s Klein bottle *Tonnetz* [16] in a manner explained in detail in Section 3. Briefly, our space  $C(1, 2, 3)$  is not a surface because the tritone intervals are shared by four 2-simplices. In Peck’s Klein bottle *Tonnetz*, each of these tritones is represented by two 1-simplices, each of which is shared by two 2-simplices, thereby forming a surface, the Klein bottle. As we show in Section 3, there are choices involved in the process of splitting the tritones in two, which lead to 8 versions of Peck’s Klein bottle *Tonnetz*, half of which are Klein bottles, and half of which are tori.

The simplicial complexes we consider may not even be connected. For example, the augmented triads form the 2-simplices of  $C(4, 4, 4)$ , since they are characterized by the fact that there are four semi-tones between each note in the chord.  $C(4, 4, 4)$  has four components,  $\{C, E, G^\sharp\}$ ,  $\{C^\sharp, F, A\}$ ,  $\{D, F^\sharp, A^\sharp\}$ , and  $\{D^\sharp, G, B\}$ ,

Connected component of $C(n_1, n_2, n_3)$	Relations on $n_1, n_2$ , and $n_3$
2-simplex	$n_1 = n_2 = n_3$
tetrahedron boundary	$n_1 = n_2 < n_3 = N/2$
cylinder	$n_1 = n_2 < n_3 \neq N/2$ or $n_1 < n_2 = n_3$ with $N$ even
Möbius band	$n_1 = n_2 < n_3 \neq N/2$ or $n_1 < n_2 = n_3$ with $N$ odd
circle of $(n_1 + n_2)/\gcd(n_1, n_2)$ tetrahedra boundaries	$n_1 + n_2 = n_3 = N/2$
torus	$n_1 < n_2 < n_3$

FIGURE 1. The classification of generalized *Tonnetze*.

each of which consists of a single 2-simplex (i.e., chord). The complete symmetry of the augmented triad means that we have only four of them in the chromatic scale, corresponding to the four disjoint 2-simplices of  $C(4, 4, 4)$ .

Another example of a non-connected generalized *Tonnetz* is  $C(3, 3, 6)$ , whose 2-simplices are the diminished triads, since the diminished triad consists of three notes which differ by three, three, and six semi-tones. The space they form,  $C(3, 3, 6)$ , can be decomposed into three disjoint copies of  $C(1, 1, 2)$ , which is a tetrahedron. Each tetrahedron contains four of the twelve possible diminished triads. For example, one of the three contains the 4 chords  $\{C, D^\sharp, F^\sharp\}$ ,  $\{F^\sharp, A, C\}$ ,  $\{A, F^\sharp, D^\sharp\}$ ,  $\{A, C, D^\sharp\}$ . The remaining diminished triads fall into two other families like this, and these families share no notes.

After these illustrations, we may now state our main theorems precisely. Let  $n_1, n_2, n_3$ , and  $N$  be positive integers such that  $n_1 + n_2 + n_3 = N$  and  $1 \leq n_1 \leq n_2 \leq n_3 < N$ . Then the connected **components** of the generalized *Tonnetz*  $C(n_1, n_2, n_3)$  are 2-simplices, tetrahedra boundaries, tori, cylinders, Möbius bands, or circles of tetrahedra boundaries (Theorem 32). Moreover, any two connected components of  $C(n_1, n_2, n_3)$  are isomorphic (Theorem 16). The simplicial complex  $C(n_1, n_2, n_3)$  is connected if and only if  $\gcd(n_1, n_2, n_3) = 1$  (Theorem 15). The complete list of possibilities is contained in Figure 1, and is proved in Sections 6.1, 6.2, and 6.3.

In Figure 2, we list all the possibilities for  $\mathbb{Z}/12$ . Note that  $\mathbb{Z}/12$  contains every possibility except the Möbius band. This is the best we could expect, since the Möbius band occurs only in scales with an odd number of notes. (See Theorem 28.)

The paper is organized as follows. We begin with definitions and examples in Section 2. Next, in Section 4, we count the number of vertices, edges, and faces in  $C(n_1, n_2, n_3)$ . Using this information we determine the Euler characteristic of  $C(n_1, n_2, n_3)$ . We use this to classify those spaces of triads which are compact, connected, orientable surfaces in Section 6. We discuss the connectedness of spaces of triads in Section 5. If the space is not connected, it must consist of disjoint, identical subspaces. This, along with homogeneity (using the operations of the  $T/I$  group), allows us to classify the remaining spaces. We use ideas from geometry, algebraic topology, homological algebra, and elementary number theory. The method of proof is explicit: we consider a general vertex (pitch class)  $a$  and all 2-simplices (triads) which contain it. The discussion of connected spaces of triads naturally falls into three cases: surfaces with boundary, surfaces without boundary, and spaces which are not surfaces.

We can see that the most like case is a torus, by the following argument. The number of possible shapes  $\{n_1, n_2, n_3\}$  in an  $N$ -note scale is  $aN^2 + \text{lower degree terms}$ . (In fact,  $a = 1/12$  but this is not important). This follows since the number of choices for each of  $n_1$  and  $n_2$  is proportional to  $N$ , and  $n_3$  is then determined by  $n_3 = N - n_1 - n_2$ . Each of the non-torus cases is determined by one or more equalities, such as  $n_1 = n_2$ . The number of triples  $\{n_1, n_2, n_3\}$  satisfying this in addition is  $bN + \text{constant}$ , for some  $b$ . Adding all the non-torus cases, we find their proportion is

$$\frac{cN + \text{constant}}{aN^2 + \text{linear} + \text{constant}}$$

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$C(n_1, n_2, n_3)$	Simplicial Complex	Representative Trichord
$C(1, 1, 10)$	cylinder	$\{0, 1, 2\} = \{C, C^\sharp, D\}$
$C(1, 2, 9)$	torus	$\{0, 1, 3\} = \{C, C^\sharp, D^\sharp\}$
$C(1, 3, 8)$	torus	$\{0, 1, 4\} = \{C, C^\sharp, E\}$
$C(1, 4, 7)$	torus	$\{0, 1, 5\} = \{C, C^\sharp, F\}$
$C(1, 5, 6)$	circle of 6 tetrahedra boundaries	$\{0, 1, 6\} = \{C, C^\sharp, F^\sharp\}$
$C(2, 2, 8)$	two disjoint cylinders	$\{0, 2, 4\} = \{C, D, E\}$
$C(2, 3, 7)$	torus	$\{0, 2, 5\} = \{C, D, F\}$
$C(2, 4, 6)$	two disjoint circles of 3 tetrahedra boundaries	$\{0, 2, 6\} = \{C, D, F^\sharp\}$
$C(2, 5, 5)$	cylinder	$\{0, 2, 7\} = \{C, D, G\}$
$C(3, 3, 6)$	three disjoint tetrahedra boundaries	$\{0, 3, 6\} = \{C, E^\flat, G^\flat\}$
$C(3, 4, 5)$	torus	$\{0, 3, 7\} = \{C, E^\flat, G\}$
$C(4, 4, 4)$	four disjoint 2-simplices	$\{0, 4, 8\} = \{C, E, G^\sharp\}$

FIGURE 2. The generalized *Tonnetze* found in  $\mathbb{Z}/12$ .

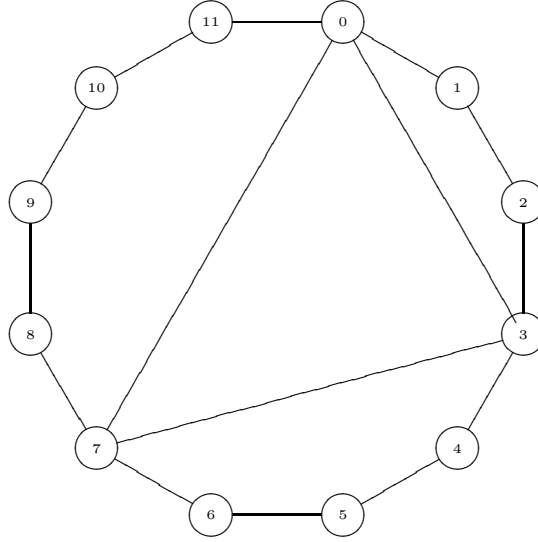


FIGURE 3. The C-minor triad  $\{0, 3, 7\}$  is a 2-simplex of  $C(3, 4, 5)$ .

which goes to 0 as  $N \rightarrow \infty$ . Hence, as  $N$  grows, the probability of *not* finding a torus decreases to zero.

In Section 3, we compare our results to those of Balzano, Cohn, Mazzola, and Peck. In the first three cases, our results are the same. Peck's Klein bottle *Tonnetze*, on the other hand, can be mapped to our spaces, but are not identical.

## 2. PRELIMINARY DEFINITIONS

In order to help visualize these complexes, we arrange the  $N$  note scale symmetrically on a circle. We number the notes in the scale from 0 to  $N - 1$  in an increasing, clockwise fashion on the perimeter of the circle. In Figure 3, we see the musical circle for  $\mathbb{Z}/12$  with a triad inscribed in it.

Let us show  $C(3, 4, 5)$  is the neo-Riemmanian *Tonnetz* of major and minor triads in more detail. This is discussed in many sources including [11], [14]. Throughout this section, we will be referring to the musical circle in Figure 3, with the  $\{0, 3, 7\}$  triad inscribed in it. Let us label pitch classes in the usual way:



FIGURE 4. Two examples of simplicial complexes

0	1	2	3	4	5	6	7	8	9	10	11
C	C <sup>#</sup>	D	D <sup>#</sup>	E	F	F <sup>#</sup>	G	G <sup>#</sup>	A	A <sup>#</sup>	B

We see that the triad we have been referring to,  $\{0, 3, 7\}$ , is the c-minor triad. Other common examples are  $C = \{0, 4, 7\}$ ,  $E = \{4, 8, 11\}$ , and  $F = \{0, 5, 9\}$ . If we construct these 2-simplices from our musical circle and attach them as in Figure 5, we see the space in Figure 6. Let  $1 \leq n_1 \leq n_2 \leq n_3$  and  $N = n_1 + n_2 + n_3$ . A type of triad in an  $N$ -note scale can be described by giving the distances  $n_1$ ,  $n_2$ , and  $n_3$  between the notes arranged cyclically. Thus, a minor triad in  $\mathbb{Z}/12$  corresponds to  $(n_1, n_2, n_3) = (3, 4, 5)$  since, for example, there are 3 half steps from C to E<sup>b</sup>, 4 half steps from E<sup>b</sup> to G, and 5 from G to C, always counting upward in pitch. Since the number of notes in our scale is  $N$ , we shall use the group  $\mathbb{Z}/N$  of integers modulo  $N$  to label the notes.

The mathematical object we use to model a space of triads is that of a simplicial complex, which is a set theoretic abstraction of polyhedral geometry. Formally, a simplicial complex consists of a finite set  $\mathcal{V}$  of *vertices* together with a set  $\mathcal{S}$  of subsets of  $\mathcal{V}$  called *simplices* with the property that any subset of a simplex is again a simplex. We call subsets in  $\mathcal{S}$  with  $n + 1$  elements  $n$ -simplices.

Each simplicial complex has a geometric realization as a polyhedron, obtained by associating to each 0-simplex  $\{v\}$  a point which we shall call  $v$ , to each 1-simplex  $\{v, w\}$  an interval from  $v$  to  $w$ , to each 2-simplex  $\{u, v, w\}$  a solid triangle bounded by the 1-simplices  $\{u, v\}$ ,  $\{u, w\}$  and  $\{v, w\}$  (so that its vertices are  $u$ ,  $v$  and  $w$ ) and so on.

The condition that subsets of simplices are themselves simplices implies that as we build up the complex, the lower dimensional simplices we need as boundaries of higher dimensional ones are already present.

For example, the simplicial complex with  $\mathcal{V} = \{0, 1, 2\}$  and  $\mathcal{S} = \{\{0\}, \{1\}, \{2\}\}$  represents the vertices of a triangle. However, the complex  $\mathcal{V} = \{0, 1, 2\}$ ,  $\mathcal{S} = \{\{0, 1\}, \{1, 2\}, \{0, 1\}, \{1, 2\}\}$  represents the vertices as well as two edges of a triangle. The geometric realization of these two examples can be seen in Figure 4.

A simplicial complex consisting of all non-empty subsets of an  $(n + 1)$  element set is called an  $n$ -simplex because its geometric realization is the simplest  $n$ -dimensional polyhedron. Any simplicial complex is the union of the simplices contained within it.

One advantage of simplicial complexes is that they allow one to study polyhedral topology by working with finite sets. In particular, *simplicial homology* is extraordinarily useful. We shall use it in Section 5.1 to determine the number of connected components in our *Tonnetze*, so we shall not be concerned with the general definition here. However, we would like to suggest one idea. Intuitively, the  $n$ -th homology measures the ‘ $n$ -dimensional holes’ in a space, so it may be a useful tool in analyses. For example, the neo-Riemannian *Tonnetz* forms a torus, so that there are two independent paths on it, out of which all others can be composed. This suggests that there are essentially two independent ways in which one can form chord progressions which start and end at the same chord without repeating themselves. Naturally, this is mere speculation, and would require careful analysis to determine whether homology groups have musical significance (see [2] for more on this topic).

We label the distances between notes in a least to greatest fashion, so  $1 \leq n_1 \leq n_2 \leq n_3$ . If we denote the number of notes in the scale by  $N$ , then  $N = n_1 + n_2 + n_3$  since this corresponds to a complete circuit of the musical circle for  $C(n_1, n_2, n_3)$ .

**Definition 1.** Let  $N$  be a positive integer and  $n_1, n_2, n_3$  be positive integers such that  $n_1 + n_2 + n_3 = N$  and  $1 \leq n_1 \leq n_2 \leq n_3 < N$ . We denote by  $C(n_1, n_2, n_3)$  the abstract simplicial complex in which

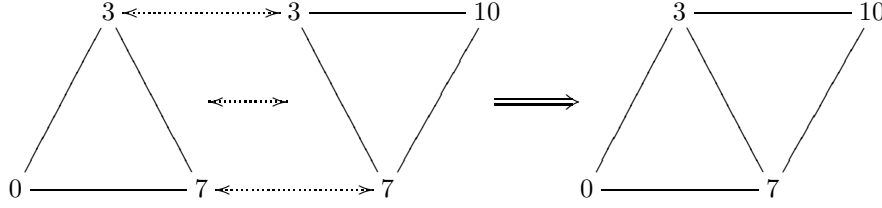


FIGURE 5. Gluing the edges together.

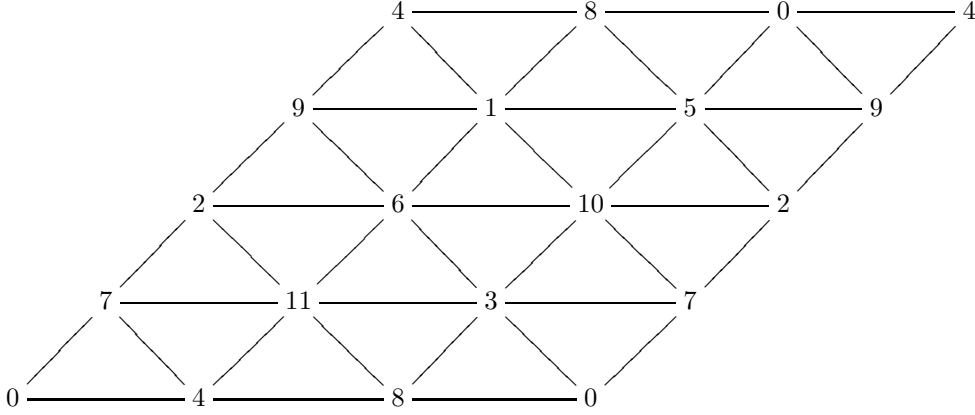


FIGURE 6. A fundamental region for  $C(3, 4, 5)$ .

- (1) the set of 0-simplices is  $\mathbb{Z}/N$ ,
- (2) the set of 1-simplices consists of all mod  $N$  translations of  $\{0, n_1\}$ ,  $\{0, n_2\}$ ,  $\{0, n_3\}$ , and
- (3) the set of 2-simplices consists of all mod  $N$  translations and inversions of  $\{0, n_1, n_1 + n_2\}$ .

In other words, the set of 2-simplices of  $C(n_1, n_2, n_3)$  is the  $T/I$ -class of  $\{0, n_1, n_1 + n_2\}$  and the 0- and 1-simplices are the vertices and edges of these 2-simplices. The geometric realization of this abstract simplicial complex, which is also denoted  $C(n_1, n_2, n_3)$ , is then a simplicial complex called a space of triads.

Let  $\mathcal{F}$  be the set of all faces,  $\mathcal{E}$  be the set of all edges, and  $\mathcal{V}$  be the set of all vertices of  $C(n_1, n_2, n_3)$ . To view  $C(n_1, n_2, n_3)$  as a space via geometric realization, take a solid triangle for each 2-simplex and join them along common edges. For example, in  $C(3, 4, 5)$ , the face  $\{0, 3, 7\}$  and the face  $\{3, 7, 10\}$  would be joined along the edge  $\{3, 7\}$  (see Figure 5).

### 3. EXAMPLES OF SPACES OF TRIADS

Let us show  $C(3, 4, 5)$  is the neo-Riemmanian *Tonnetz* of major and minor triads in more detail. In Figure 6, we have shown each 2-simplex (face) exactly once. Edges along the boundary of our drawing occur twice and so are joined in the space  $C(3, 4, 5)$ . The top and bottom vertices begin to repeat, and we see that these edges are actually equal. Similarly, the left and right sides are equal. Thus, we can take the top edge and glue it on to the bottom. Similarly gluing the left and right edges, we see that we have a torus. Hence, the space  $C(3, 4, 5)$ , of major and minor triads in  $\mathbb{Z}/12$ , is a torus.

Mazzola's harmonic strip [10, pp. 321-322] is another interesting example. The harmonic strip has vertices  $\mathbb{Z}/7$ . Specifically, it has seven 0-simplices, fourteen 1-simplices, and seven 2-simplices. By Theorem 12, this space must either be  $C(1, 1, 5)$ ,  $C(1, 3, 3)$  or  $C(2, 2, 3)$ . Theorem 28 implies this space will be a Möbius band, in agreement with Mazzola's findings.

In [9], Balzano discusses 12-fold and microtonal pitch systems. He describes several examples, including different spaces of triads with 20, 30, and 42 notes [9, pp. 77-80]. These spaces are  $C(4, 5, 11)$ ,  $C(5, 6, 19)$ ,

and  $C(6, 7, 29)$  respectively. He also discusses the analogs of the Riemannian *Tonnetz* generated by thirds and fifths from the diatonic scale in the more general  $C(n, n+1, n^2 - n - 1)$ . [9, p. 75]. Theorem 23 implies that this space will always be a torus.

Cohn's work in [15] concerning the realizations of *Parsimonious Tonnetz* also requires mention. In [15, Sec. 2.3], Cohn computes the requirements on  $N$ ,  $n_1$ , and  $n_2$  such that they lead to 'optimally parsimonious voice-leading' under PLR-family operations. He proves that the only spaces which support parsimonious voice-leading are the  $C(n, n+1, n+2)$ . By Theorem 23, these *Parsimonious Tonnetz* are tori when  $n > 1$  (when  $n = 1$ , a more exotic space occurs).

Cohn also describes the *LPR* loops of the Riemannian *Tonnetz* in [15]. An *LPR* loop consists of a triad and all other triads obtained by applying the  $R$ ,  $P$ , and  $L$  transformations to it, in that order, until returning to the original triad. This is of interest in connection with parsimonious voice-leading. In [15, Sec. 3.6], Cohn discusses the placement of the six triads in an *LPR* loop around a given vertex, as shown in our Figure 9. The cyclical ordering of the triads containing a particular pitch class by means of an *LPR* loop play a significant role in determining which generalized *Tonnetz* form surfaces. This is not possible in every case: specifically when  $C(n_1, n_2, n_3)$  is a cylinder, Möbius band, or circle of tetrahedra boundaries. In Section 6.2 (notably Figures 13 and 14), we discuss the cases in which a full *LPR* loop cannot be made around every vertex. In these cases, though, the  $P$ ,  $L$ , and  $R$  operations do not behave as they do classically. For example, if  $n_1 = n_2$ , then  $P$  becomes the identity map.

One space which does not occur as one of our generalized *Tonnetz* is the Klein bottle. Peck, on the other hand, discovers Klein bottle *Tonnetz* in his analysis of musical spaces [16]. Peck's Klein bottle *Tonnetz* occur in those cases in which we find circles of tetrahedra boundaries. Theorem 31 implies that these are exactly the cases in which the chords contain tritones (intervals of length  $N/2$ ). In these cases, our *Tonnetz* are quotients of his, obtained by collapsing certain pairs of edges. In Peck's Klein bottle *Tonnetz*, there are two edges corresponding to each tritone, while in ours, these are considered to be a single edge. This allows Peck's *Tonnetz* to be surfaces, while ours are singular along these edges, since four distinct 2-simplices are joined at them.

Peck [16, Fig. 20] shows the pitch class *Tonnetz* consisting of the even pitch classes in  $C(2, 4, 6)$ . We reproduce his figure here as Figure 7. Examining it, the relation between the rows is key. Moving from one row to the next is done by applying an operation on triads which fixes the tritone and moves the third note. In *Tonnetz* of chords which do not contain a tritone, there is a unique operation in the *PLR* group which does this. However, when the fixed interval is a tritone, there are four triads which share it, so there are three other triads to which the triad could move. In these *Tonnetz*, triads fall into two types. The transpositions of  $\{0, n_1, N/2\}$  are distinct from their inversions, which are transpositions of  $\{0, n_2, N/2\}$ . (Recall that  $n_1 + n_2 = N/2$  in the *Tonnetz* under consideration.)

Of the four operations on triads which fix the tritone, two will exchange these two types, just as the  $P$ ,  $L$ , and  $R$  operations do. Let us call these  $S$  and  $F$ , where  $S$  moves the third note to a note in the *same* half of the musical circle, and  $F$  *flips* it to the appropriate note in the other half of the musical circle. For example,  $S(\{0, n_1, N/2\}) = \{0, n_2, N/2\}$ , while  $F(\{0, n_1, N/2\}) = \{0, -n_1, N/2\} = \{0, n_2 + n_3, N/2\}$ . The composite  $SF = T_{N/2}$ .

Examining Peck's Figure 20 (Figure 7), we see that the transition from row I to row II is done by applying  $F$ , since  $F(\{2, 4, 8\}) = \{2, 0, 8\}$  and  $F(\{2, 10, 8\}) = \{2, 6, 8\}$ . The transition from row II to row III is by  $S$ :  $S(\{0, 2, 6\}) = \{0, 4, 6\}$  and  $S(\{0, 8, 6\}) = \{0, 10, 6\}$ . To make the transition from row III back to row I we must decide whether to apply  $S$  or  $F$ . If we apply  $S$ , we get Peck's Klein bottle *Tonnetz*, as shown in Figure 8. We have labelled the 2-simplices in the top and bottom rows of this diagram  $A$ ,  $B$ ,  $C$  and  $D$ , to make evident the reversal which gives us a Klein bottle.

Had we chosen to join row III to row I by the operation  $F$  instead, we would obtain the other *Tonnetz* shown in Figure 8. Here, we evidently get a torus. Choosing either  $S$  or  $F$  to join each row to the next gives us eight choices in assembling a *Tonnetz* from the 2-simplices of the even component of  $C(2, 4, 6)$ . We could call them  $SSS$ ,  $SSF$ ,  $SFS$ ,  $\dots$ ,  $FFF$ , with Peck's Klein bottle *Tonnetz* corresponding to  $FSS$  in the ordering of simplices used in Peck's Figure 20 and our Figures 7 and 8. Half of these are tori and half are Klein bottles, depending upon the number of flips  $F$  used to assemble them. If this number is even, we get a torus. If it is odd, we get a Klein bottle.

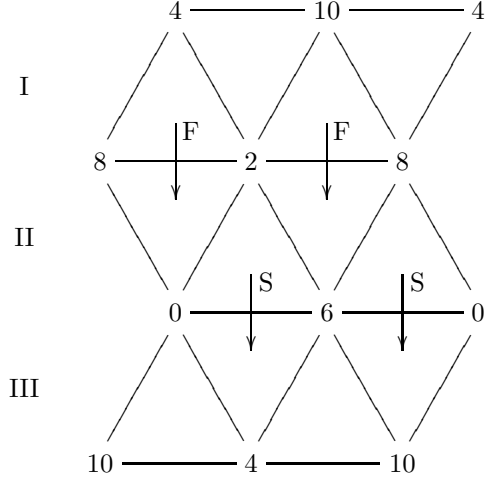


FIGURE 7. Peck's Klein bottle *Tonnetz* [16, Fig. 20]

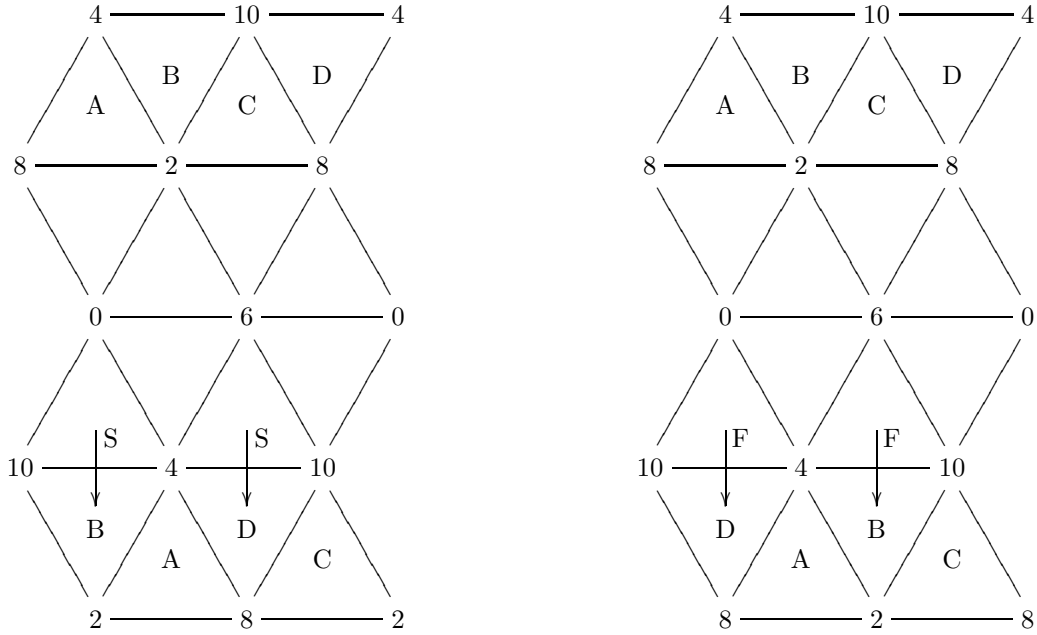


FIGURE 8. Peck's Klein bottle *Tonnetz* and a torus variant

In our *Tonnetz*  $C(2, 4, 6)$ , each row is formed into a tetrahedron, as in Figure 11, since each tritone corresponds to a single edge. These then join one another in a circular fashion, as shown in Figure 15. Each of the analogs of Peck's Klein bottle *Tonnetz*, distinguished by the sequences  $SSS$  through  $FFF$ , map to  $C(2, 4, 6)$  by collapsing the two copies of each tritone to one.

As an interesting side note here, the analog of the *PLR* group will be the same for all these *Tonnetze*, since their sets of 2-simplices are the same. The differences between them arise from the manner in which these 2-simplices are joined. The analog of the *PLR* group cannot act simply transitively on the 2-simplices, since there are two components and  $P$ ,  $L$ , and  $R$  preserve the components. To get a simply transitive group action, one must use the generalized contextual group of Fiore and Satyendra [17].

Another interesting fact, discussed in the introduction and proven in Theorem 16, is that if the edge lengths share a common divisor, then the space will consist of independent, identical components (Corollary 15). Peck's Fig. 20 and 21 illustrate this fact. In these examples, Peck considers  $C(2, 4, 6)$ , which he shows consists of two copies of  $C(1, 2, 3)$ , one containing the even pitch classes ([16, Fig. 20]) and one containing the odd ([16, Fig. 21]).

#### 4. THE EULER CHARACTERISTIC

The first step in our classification of spaces of triads is a computation of the Euler characteristic. The Euler characteristic is a geometric invariant of spaces which distinguishes the compact, orientable surfaces without boundary. That is, two compact orientable surfaces without boundary which have the same Euler characteristic are topologically equivalent. For example, every torus, e.g. the neo-Riemannian *Tonnetz*, has Euler characteristic zero. When  $C(n_1, n_2, n_3)$  is non-orientable (like the Möbius band), or has a boundary, or is not a surface, more work will be necessary to complete our classification of spaces of triads.

The Euler characteristic of a finite simplicial complex is determined by counting the  $i$ -simplices for each  $i$ . While the number of vertices in  $C(n_1, n_2, n_3)$  is simply  $N = n_1 + n_2 + n_3$ , computing the number of edges and faces requires more careful counting. We recall the definition of the Euler characteristic and then proceed with finding the number of edges and faces in our simplicial complex  $C(n_1, n_2, n_3)$ .

**Definition 2.** *The Euler characteristic,  $\chi$ , of a simplicial complex is:*

$$\chi = \sum_{i \geq 0} (-1)^i n_i$$

where  $n_i$  is the number of  $i$ -simplices. For a 2-dimensional complex, this reduces to

$$\chi = V - E + F$$

where  $V$  is the number of vertices,  $E$  is the number of edges, and  $F$  is the number of faces.

We make the following definitions to simplify the counting of the number of edges.

**Definition 3.** *Suppose  $1 \leq n_1 \leq n_2 \leq n_3 < N$  are integers and  $N = n_1 + n_2 + n_3$ .*

$$\begin{aligned} X &= \{\{k, k + n_1\} \mid k \in \mathbb{Z}/N\} \\ Y &= \{\{k, k + n_2\} \mid k \in \mathbb{Z}/N\} \\ Z &= \{\{k, k + n_3\} \mid k \in \mathbb{Z}/N\} \end{aligned}$$

Clearly,  $X$  contains all edges of  $C(n_1, n_2, n_3)$  which have length  $n_1$ ,  $Y$  contains those of length  $n_2$ , and  $Z$  contains those of length  $n_3$ . Thus, the size of the set  $X$  is the number of edges of length  $n_1$ , and similarly for  $Y$  and  $Z$ .

We start with some useful lemmas.

**Lemma 4.** *Suppose  $1 \leq n_1 \leq n_2 \leq n_3 < N$  are integers and  $N = n_1 + n_2 + n_3$ . Then  $n_1 < N/2$  and  $n_2 < N/2$ .*

*Proof.*  $2n_1 \leq 2n_2 \leq n_2 + n_3 < n_1 + n_2 + n_3 = N$ . □

**Lemma 5.** *Suppose  $1 \leq n_1 \leq n_2 \leq n_3 < N$  are integers and  $N = n_1 + n_2 + n_3$ . Then  $n_3 = n_1 + n_2$  if and only if  $n_3 = N/2$ .*

*Proof.* By the definition of  $N$ , it follows that  $n_1 + n_2 + n_3 = N = 2n_3$  if and only if  $n_1 + n_2 = n_3$ . □

**Lemma 6.** *Suppose  $1 \leq n_1 \leq n_2 \leq n_3 < N$  are integers and  $N = n_1 + n_2 + n_3$ . The sets  $X$  and  $Y$  each have  $N$  elements. The number of elements in  $Z$  is*

$$|Z| = \begin{cases} N & \text{if } n_3 \neq n_1 + n_2 \\ N/2 & \text{if } n_3 = n_1 + n_2 \end{cases}$$



*Proof.* Consider the list

$$\{0, n_1\}, \{1, 1 + n_1\}, \dots, \{(N-1), (N-1) + n_1\}.$$

Suppose  $\{i, i + n_1\} = \{j, j + n_1\}$ . Then  $i = j$  or  $i = j + n_1$ . If  $i = j$ , then  $i + n_1 = j + n_1$  and  $\{i, i + n_1\}$  and  $\{j, j + n_1\}$  are the same entry in the list. If, on the other hand,  $i = j + n_1$ , then  $j = i + n_1$  and  $i = i + n_1 + n_1$ , so  $2n_1 = 0$  in  $\mathbb{Z}/N$ . But this contradicts  $n_1 < N/2$  in Lemma 4. Thus  $i = j$  is the only possibility, and the list has  $N$  elements, so  $|X| = N$ . The proof of  $|Y| = N$  is similar, since  $n_2 < N/2$  by Lemma 4.

The proof of  $|Z| = N$  in the case  $n_3 \neq n_1 + n_2$  is also similar by Lemma 5.

To show  $|Z| = N/2$  in the case  $n_3 = n_1 + n_2$ , note that  $N = 2n_3$  by Lemma 5. Then the sets  $\{k, k + n_3\}$  are all different for  $0 \leq k < n_3$ , since each set  $\{k, k + n_3\}$  contains only one of  $0 \leq k < n_3$ . Then for  $n_3 \leq k < N$ , we have

$$\{k, k + n_3\} = \{n_3 + i, n_3 + i + n_3\} = \{i, i + n_3\}$$

for some  $0 \leq i < n_3$ , so we are back to the first half of the list.  $\square$

**Theorem 7.** *The number of edges  $|\mathcal{E}|$  of the simplicial complex  $C(n_1, n_2, n_3)$  is given by the following chart:*

Cases	$ X $	$ Y $	$ Z $	$ X \cup Y \cup Z $
$n_1 < n_2 < n_3$ and $n_3 \neq n_1 + n_2$	$N$	$N$	$N$	$3N$
$n_1 < n_2 < n_3$ and $n_3 = n_1 + n_2$	$N$	$N$	$N/2$	$5N/2$
$n_1 = n_2 < n_3$ and $n_3 \neq n_1 + n_2$	$N$	$N$	$N$	$2N$
$n_1 = n_2 < n_3$ and $n_3 = n_1 + n_2$	$N$	$N$	$N/2$	$3N/2$
$n_1 < n_2 = n_3$	$N$	$N$	$N$	$2N$
$n_1 = n_2 = n_3$	$N$	$N$	$N$	$N$

*Proof.* In the first and second rows of the table, we see that the number of edges is  $|X| + |Y| + |Z|$ . In the third and fourth rows, the number of edges is actually  $|X| + |Z|$ , since  $X = Y$ , and we avoid double counting. In the fifth row, we see that  $Y = Z$ , and hence the number of edges is only  $|X| + |Y|$ . In the last row, since all three sets are equal, we only count one of them to avoid triple counting.  $\square$

Now that we have counted the edges of  $C(n_1, n_2, n_3)$ , we count the faces. Let us call the vertex between the edges of lengths  $n_3$  and  $n_1$  the *basepoint* of the 2-simplex. Recall that  $\mathcal{F}$  is the set of all faces of  $C(n_1, n_2, n_3)$ .

**Lemma 8.** *Elements of  $\mathcal{F}$  are of the form*

$$\{\{k, k + n_1, k + n_1 + n_2\} \mid k \in \mathbb{Z}/N\}$$

or

$$\{\{k, k - n_1, k - n_1 - n_2\} \mid k \in \mathbb{Z}/N\}.$$

*Proof.* By Definition 8, the 2-simplices are the translations and inversions of  $\{0, n_1, n_1 + n_2\}$ . Let us call these *Type I* and *Type II*, respectively, with the understanding that, in exceptional circumstances, these types might overlap. Clearly, the translations are those of the form  $\{k, k + n_1, k + n_1 + n_2\}$ . Just as clearly, the inversion  $I_k(i) = k - i$  produces those of the form  $\{k, k - n_1, k - n_1 - n_2\}$ .  $\square$

In the case of  $C(3, 4, 5)$ , the Type I simplices are the minor triads and the Type II simplices are the major triads. However, in more symmetrical situations, these types may not be distinct.

**Proposition 9.** *If  $n_1 = n_2 = n_3$ , then  $|\mathcal{F}| = N/3$ .*

*Proof.* Clearly,  $N = 3n_1$ , and all 2-simplices have the form  $\{k, k + n_1, k + 2n_1\}$ . This is simply the coset  $k + \langle n_1 \rangle$  in  $\mathbb{Z}/N$ , so there are  $N/3$  of them.  $\square$

**Proposition 10.** *If  $n_1 < n_2 < n_3$ , then  $|\mathcal{F}| = 2N$ .*

*Proof.* Let us start by calculating the number of Type I 2-simplices. It is obvious that the vertex between the edges of length  $n_3$  and  $n_1$  selects a unique, well-defined value of  $k$  which provides an inverse to  $k \mapsto \{k, k + n_1, k + n_1 + n_2\}$ , so that there are  $N$  2-simplices of Type I. Similarly there are  $N$  2-simplices of Type II. No 2-simplex is of both Types simultaneously. To see this, suppose we have a face which is of both Types I and II. It must contain an interval  $\{a, a + n_1\}$ , so we let  $\{a, a + n_1, b\}$  be of Type I and  $\{a, a + n_1, c\}$  be of Type II, and suppose they are equal. Since the first is Type I,  $b$  must be  $a + n_1 + n_2$ . This follows from the inequality  $n_1 < n_2 < n_3$ , which implies that there is a unique edge of each length  $n_1$ ,  $n_2$  and  $n_3$ . Similarly,  $c$  must be  $a - n_2$  since the basepoint of this chord is  $a + n_1$ , so that it can be written  $\{a + n_1, (a + n_1) - n_1, c = (a + n_1) - n_1 - n_2\}$ . For these two triads to be equal,  $b = c$ , that is  $a - n_2 = a + n_1 + n_2$ , modulo  $N$ . Hence,  $0 = n_1 + 2n_2$ , modulo  $N$ . Lemma 4 implies that  $0 < n_1 + 2n_2 < 3N/2$ . The only number congruent to 0 modulo  $N$  in this interval is  $N$  itself, so we conclude that  $n_1 + 2n_2 = N = n_1 + n_2 + n_3$ . But this implies  $n_2 = n_3$ , contradicting our hypothesis. Hence our faces (2-simplices) have a well-defined Type, either I or II, when  $n_1 < n_2 < n_3$ .  $\square$

**Proposition 11.** *If  $n_1 = n_2 < n_3$  or  $n_1 < n_2 = n_3$ , then  $|\mathcal{F}| = N$ .*

*Proof.* Suppose  $n_1 = n_2 < n_3$ . In this case, the Type I triads can be written  $\{k, k + n_1, k + 2n_1\}$ . If we let  $j = k + 2n_1$  then the Type II triad  $\{j, j - n_1, j - 2n_1\}$  is exactly the same as the Type I triad  $\{k + 2n_1, k + n_1, k\}$ , so that the two Types coincide. Seen another way, there are two possible basepoints to a triad, since there are two edges of length  $n_1$ , and the triad is of Type I with respect to one of them and of Type II with respect to the other. Thus, the number of faces is exactly the number of Type I triads, and these are in one to one correspondence with the vertices by taking the (Type I) basepoint of the triad.

Exactly the same argument works in the case  $n_1 < n_2 = n_3$ .  $\square$

Now that we have calculated the number of vertices, edges, and faces for any simplicial complex  $C(n_1, n_2, n_3)$ , we can calculate the Euler characteristic.

**Theorem 12.** *The Euler characteristic,  $\chi$ , of  $C(n_1, n_2, n_3)$  is given by the following chart:*

Cases	$ \mathcal{V} $	$ \mathcal{E} $	$ \mathcal{F} $	$\chi$
$n_1 < n_2 < n_3$ and $n_3 \neq n_1 + n_2$	$N$	$3N$	$2N$	0
$n_1 < n_2 < n_3$ and $n_3 = n_1 + n_2$	$N$	$5N/2$	$2N$	$N/2$
$n_1 = n_2 < n_3$ and $n_3 \neq n_1 + n_2$	$N$	$2N$	$N$	0
$n_1 = n_2 < n_3$ and $n_3 = n_1 + n_2$	$N$	$3N/2$	$N$	$N/2$
$n_1 < n_2 = n_3$	$N$	$2N$	$N$	0
$n_1 = n_2 = n_3$	$N$	$N$	$N/3$	$N/3$

*Proof.* The  $\mathcal{V}$  column is clear. The  $\mathcal{E}$  column is proved in Theorem 7, and the  $\mathcal{F}$  column is proved in Lemma 8 and Propositions 9-11.  $\square$

For further emphasis, let us check our work in Section 3. In the chromatic scale, we saw that the major and minor triads,  $C(3, 4, 5)$ , form a torus. Using the chart above, we see that  $C(3, 4, 5)$  is in the first row, so must have Euler characteristic 0. In fact, the Euler characteristic is indeed 0, as expected from elementary topology.

## 5. TOPOLOGICAL PROPERTIES

We now focus our attention on some basic topological properties of the spaces  $C(n_1, n_2, n_3)$ . Concepts like path-connectedness and homogeneity are discussed here, and these types of topological properties will prove very useful in our classification of  $C(n_1, n_2, n_3)$  as a space.

**5.1. Connectedness.** To determine whether  $C(n_1, n_2, n_3)$  is connected or not, and if not, how many components it has, we use algebraic topology and elementary number theory.

Let  $H_n(X; R)$  denote the  $n$ -th homology group of a space  $X$  over a ring  $R$ . Let us recall some basic facts from algebraic topology [13]. First, it is well known that  $H_0(X; R) = R^d$ , where  $d$  is the number of connected components of  $X$ . (Recall that connected components and path-components are the same for a

simplicial complex, so we may abbreviate this to *component* without ambiguity.) This allows us to compute the number of components of the simplicial complex  $C(n_1, n_2, n_3)$  from its 0-th homology. In the following discussion, consider the free abelian group  $\mathbb{Z}[\mathcal{V}]$  generated by the vertices  $\mathcal{V} = \mathbb{Z}/N$  of  $C(n_1, n_2, n_3)$ . This is the group of 0-chains or 0-cycles (since every 0-chain is a cycle). Thus, we see

$$H_0(X; \mathbb{Z}) = \frac{Z_0(X)}{B_0(X)} \cong \frac{\mathbb{Z}[\mathcal{V}]}{\sim}$$

where  $\sim$  is given by all linear combinations of boundaries of edges. These linear combinations simply group the vertices into groups according to the component they lie in. For example, consider the linear combination  $\partial(\{0, 1\} + \{1, 2\}) = \{1\} - \{0\} + \{2\} - \{1\} = \{2\} - \{0\}$ . This calculation reflects the fact that, since there exist edges from  $\{0\}$  to  $\{1\}$  and  $\{1\}$  to  $\{2\}$ , there exists a path from  $\{0\}$  to  $\{2\}$ , as seen in Figure 4.

We now recall without proof a basic proposition concerning greatest common divisors from elementary number theory.

**Proposition 13.** *If  $m_1, \dots, m_p$  are integers, then the subgroup of  $\mathbb{Z}$  generated by  $m_1, \dots, m_p$  is  $d\mathbb{Z}$  where  $d$  is the greatest common divisor of  $m_1, \dots, m_p$ .*

**Theorem 14.**  $H_0(C(n_1, n_2, n_3); \mathbb{Z}) = \mathbb{Z}^{\gcd(n_1, n_2, n_3)}$ .

*Proof.* From the discussion above, we conclude that  $H_0 = \mathbb{Z}[\mathcal{V}/\sim]$ , where  $\sim$  means ‘is connected by a sequence of edges’. Now  $\mathcal{V} = \mathbb{Z}/N$  and the relation  $\{k\} \sim \{l\}$  is the equivalence relation generated by  $k$  differs from  $l$  by either  $n_1$ ,  $n_2$ , or  $n_3$ . Hence,

$$\mathcal{V}/\sim \cong (\mathbb{Z}/N)/\langle n_1, n_2, n_3 \rangle \cong \mathbb{Z}/\langle n_1, n_2, n_3 \rangle$$

since  $N = n_1 + n_2 + n_3$  and by the third isomorphism theorem (see [6]). By Proposition 13, we see that

$$\mathbb{Z}/\langle n_1, n_2, n_3 \rangle \cong \mathbb{Z}/\gcd(n_1, n_2, n_3).$$

Hence, we see that  $H_0(C(n_1, n_2, n_3); \mathbb{Z}) = \mathbb{Z}^{\gcd(n_1, n_2, n_3)}$ . □

**Corollary 15.** *The space  $C(n_1, n_2, n_3)$  is connected if and only if  $\gcd(n_1, n_2, n_3) = 1$ .*

*Proof.* This follows immediately from the above theorem, noting that  $H_0$  detects the number of connected components of a simplicial complex. □

As an example, let us apply the above corollary to  $C(3, 4, 5)$ . Since  $\gcd(3, 4, 5) = 1$ , Corollary 15 implies that the space  $C(3, 4, 5)$  is connected. This is just confirmation of what we already knew: the torus is connected.

**Theorem 16.** *The simplicial complex  $C(dn_1, dn_2, dn_3)$  is the disjoint union of  $C(n_1, n_2, n_3)$ . Specifically,*

$$C(dn_1, dn_2, dn_3) = \coprod_d C(n_1, n_2, n_3).$$

*Proof.* A 2-simplex with edges of lengths  $dn_1$ ,  $dn_2$ , and  $dn_3$ , must connect vertices which differ by multiples of  $d$ . Hence, the vertices in any simplex must lie in a coset of  $\langle d \rangle$  in  $\mathbb{Z}/N$ . Thus, the set of all simplices can be decomposed into a disjoint union over these cosets. □

Corollary 15 shows that if the  $\gcd(n_1, n_2, n_3) = 1$ , then  $C(n_1, n_2, n_3)$  cannot be decomposed further. Hence, if  $\gcd(n_1, n_2, n_3) = 1$ , Theorem 16 gives the decomposition of  $C(dn_1, dn_2, dn_3)$  into its connected components.

Since the analogues of the neo-Riemannian  $P$ ,  $L$ , and  $R$  operations flip triads across edges, they cannot take a triad in one component to a triad in a different component. Thus, when  $C(n_1, n_2, n_3)$  is disconnected, the group generated by the  $P$ ,  $L$ , and  $R$  analogues cannot act simply transitively on the triads in  $C(n_1, n_2, n_3)$ , and this group is a proper subgroup of Fiore–Satyendra’s generalized contextual group associated to  $\{0, n_1, n_1 + n_2\}$  in [17]. For example, in  $C(2, 4, 6)$ , no combination of the  $P$ ,  $L$ , and  $R$  analogues takes  $\{0, 2, 6\}$  to  $\{1, 3, 7\}$ , though the contextual operation  $Q_1$  certainly does.

**5.2. Homogeneity.** In general, homogeneity means that every point is the same as every other in some appropriate sense. We say  $C(n_1, n_2, n_3)$  is *homogeneous for  $n$ -simplices* if for any two  $n$ -simplices  $\sigma$  and  $\sigma'$  there is an automorphism of the simplicial complex  $C(n_1, n_2, n_3)$  which maps  $\sigma$  to  $\sigma'$ . For our simplicial complexes, we can ask for such homogeneity among vertices, among edges, or among 2-simplices. Since  $T_{j-i}(i) = j$ , we clearly have homogeneity among vertices. This means the analysis of any one vertex suffices to describe them all.

We also have homogeneity among 2-simplices using the full  $T/I$  group. For any two 2-simplices of the same Type, there is a translation taking one to the other. Inversion, on the other hand, converts a simplex from one Type to the other. Hence, given any two 2-simplices  $A$  and  $B$ , there is a member of the full  $T/I$  group which sends  $A$  to  $B$ . Therefore, our analysis of any one 2-simplex suffices to describe all 2-simplices.

However, we do not have homogeneity among the 1-simplices. For example, cylinders and Möbius bands (e.g., Mazzola's Harmonic strip) have boundary edges and interior edges, which cannot be sent to one another by an invertible map of simplicial complexes, since such a map will preserve the number of 2-simplices an edge is contained in. Similarly, in the circle of tetrahedra boundaries, there are edges (corresponding to tritones) which lie in four 2-simplices and edges which lie in only two 2-simplices, so there is no invertible simplicial map which can send the first sort to the second, or vice versa.

## 6. CLASSIFICATION OF $C(n_1, n_2, n_3)$ AS A SPACE.

We are now ready to classify all of the possible spaces of triads. We begin by considering the number of simplices which contain a given dege, since this allows us to distinguish between surfaces with boundary, surfaces without boundary, and non-surfaces by Theorem 21. Lemma 19 shows that an edge can be contained in only one or two 2-simplices unless  $n_3 = N/2$ , since Lemma 4 eliminates the possibility that  $n_1$  or  $n_2$  can be  $N/2$ . When  $n_3 \neq N/2$ , we obtain a surface with or without boundary, in Theorems 23 through 28. Finally, Theorem 31 handles the case when  $n_3 = N/2$ , in which  $C(n_1, n_2, n_3)$  is not a surface. Note that Lemma 5 says that  $n_3 = N/2$  is equivalent to  $n_3 = n_1 + n_2$ .

**6.1. When is  $C(n_1, n_2, n_3)$  a surface without boundary?** The first step in our classification of  $C(n_1, n_2, n_3)$  as a space begins with surfaces without boundary, or just surfaces. We define this notion, then investigate the pertinent cases. We prove if  $C(n_1, n_2, n_3)$  is a surface without boundary, then it is a disjoint union of tori or a disjoint union of spheres.

**Definition 17.** A surface without boundary (or just surface) is a second countable Hausdorff, topological space in which every point has an open neighborhood homeomorphic to some open subset of  $\mathbb{R}^2$ .

These include the sphere, which is topologically equivalent to the boundary of a tetrahedron, and the torus. The requirement that a surface be second countable and Hausdorff serves to eliminate pathological examples.

**Definition 18.** A surface with boundary is a second countable Hausdorff, topological space whose points have neighborhoods which are homeomorphic to open subsets of  $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ .

Points which have an open neighborhood homeomorphic to an open subset of  $\mathbb{R}^2$  are called *interior points*. The other points are called *boundary points*. Examples include the 2-simplex and rectangle, whose boundary points are their perimeters, the cylinder, whose boundary consists of two circles, and the Möbius band, whose boundary is a single circle.

There are two key properties required for a 2-dimensional simplicial complex to be a surface: (i) each edge must lie in exactly two 2-simplices, and (ii) at each vertex, the 2-simplices containing the vertex can be cyclically ordered (sharing an edge with each neighbor). To be a surface with boundary: (i) each edge must be contained in one or two 2-simplices, and (ii) at each vertex, the 2-simplices containing the vertex can be placed in either a cyclic order (in which case the vertex is an interior point) or a linear order (in which case the vertex is a boundary point).

We begin by analyzing the first property of a surface noted above. We prove the following lemma in order to make the identification of edges simpler. Let  $\{i, j, k\} = \{1, 2, 3\}$ , so that  $\{n_i, n_j, n_k\} = \{n_1, n_2, n_3\}$ . Recall that we assume  $n_1 + n_2 + n_3 = N$  and  $1 \leq n_1 \leq n_2 \leq n_3 < N$ .

**Lemma 19.** *If  $n_i \neq N/2$ , then each edge of length  $n_i$  in  $C(n_1, n_2, n_3)$  is contained in exactly one or two 2-simplices.*

*Proof.* Since  $n_i \neq N/2$ , an edge of length  $n_i$  can be written uniquely as  $\{a, a + n_i\}$ . The 2-simplices containing this edge must have vertices  $\{a, a + n_i, b\}$ . We see that either  $b = a + n_i + n_j$  or  $b = a + n_i + n_k$ . Hence, there exist two 2-simplices containing this edge if  $n_j \neq n_k$  and one such 2-simplex if  $n_j = n_k$ .  $\square$

**Lemma 20.** *If  $n_i = N/2$ , then each edge of length  $n_i$  in  $C(n_1, n_2, n_3)$  is contained in exactly two or four 2-simplices.*

*Proof.* We see that  $i = 3$  (by Lemma 4) and  $\{a, a + n_3\}$  no longer determines which vertex is  $a$ . The edge divides the musical circle in half and the third vertex of the 2-simplex containing it can be in either half. If  $n_1 \neq n_2$ , there are two choices on either side of the circle, yielding a total of four 2-simplices which contain the edge. On the other hand, if  $n_1 = n_2$ , then there is a unique choice on each side, giving only two such 2-simplices.  $\square$

The above calculations allow us to determine when  $C(n_1, n_2, n_3)$  is a surface with or without boundary. If an edge is contained in more than two 2-simplices, then the space will have singularities, and is not a surface. If an edge is only contained in one 2-simplex, then the space will have a boundary.

**Theorem 21.** *The number of 2-simplices in  $C(n_1, n_2, n_3)$  containing an edge of length  $n_i$  is determined by the following chart:*

	$n_i \neq N/2$	$n_i = N/2$
$n_j = n_k$	1	2
$n_j \neq n_k$	2	4

*Proof.* This follows directly from Lemmas 19 and 20 and their proofs.  $\square$

The other key property of a surface without boundary is the ordering of the 2-simplices in a cyclic fashion around each vertex. In order to investigate this criterion, we list all the possible 2-simplices in  $C(n_1, n_2, n_3)$  which could contain the vertex  $\{a\}$ .

- (1)  $\{a, a + n_1, a + n_1 + n_2\}$
- (2)  $\{a, a + n_1, a + n_1 + n_3\}$
- (3)  $\{a, a + n_2, a + n_1 + n_2\}$
- (4)  $\{a, a + n_2, a + n_2 + n_3\}$
- (5)  $\{a, a + n_3, a + n_1 + n_3\}$
- (6)  $\{a, a + n_3, a + n_2 + n_3\}$

When we have additional relations among the numbers  $n_1, n_2$ , and  $n_3$ , we expect some elements of this list to be redundant. Observe that we can arrange the 2-simplices in this list cyclically as shown in Figure 9. In this picture, we see each of the six 2-simplices in the list above placed around the vertex  $\{a\}$ . Hence, if all six 2-simplices are distinct, the cyclic ordering of 2-simplices containing each vertex can be accomplished. (Figures 11 and 13 show what can happen when they are not all distinct.)

We will now show that all the spaces  $C(n_1, n_2, n_3)$  which are surfaces without boundary are orientable. (The cyclical ordering does not guarantee this.) The Euler characteristic (which we have already computed) then completely determines the topological type of the surface. To do this we will assign consistent orientations to all 2-simplices in  $C(n_1, n_2, n_3)$ . We begin by noting a relation between adjacent 2-simplices.

**Lemma 22.** *If  $n_1 < n_2 < n_3 \neq N/2$ , then 2-simplices which share an edge must be of opposite Type.*

*Proof.* A 2-simplex is considered to be Type I if the edge lengths have cyclic order  $n_1, n_2, n_3$  when read in clockwise order. A 2-simplex is considered to be Type II if the edge lengths have cyclic order  $n_1, n_3, n_2$  when read in clockwise order. Since  $n_3 \neq N/2$ , each edge of length  $n_i$  can be written uniquely as  $\{a, a + n_i\}$ . By Lemma 19, there are two 2-simplices sharing this edge,  $\{a, a + n_i, a + n_i + n_j\}$  and  $\{a, a + n_i, a + n_i + n_k\}$ . However, this first 2-simplex has edge lengths  $(n_i, n_j, n_k)$  and the second has  $(n_i, n_k, n_j)$ . Hence, whenever we have a 2-simplex of either Type, any 2-simplex sharing an edge with it will be of the opposite Type.  $\square$

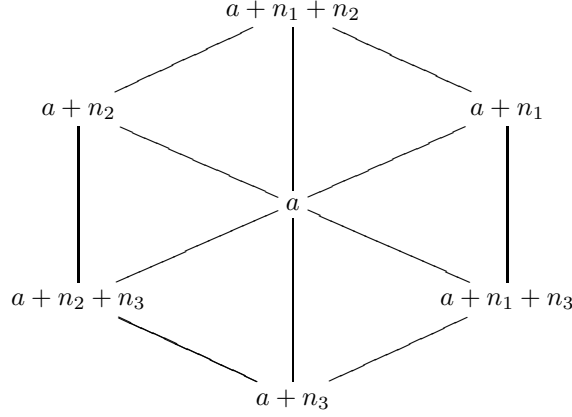


FIGURE 9. The cyclic placement of 2-simplices around the vertex  $a$ .

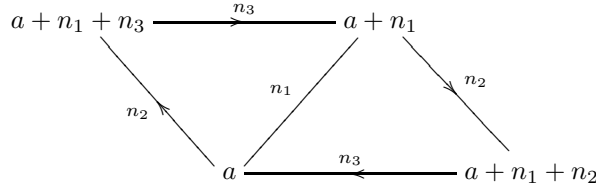


FIGURE 10. The orientation of our 2-simplices. Note the cancellation along the common edge.

This generalizes the duality of major and minor triads mentioned earlier in Lemma 8. On the other hand, if any two edge lengths are equal, then the two types coincide, as shown in the proofs of Propositions 9 and 11.

**Theorem 23.** *If  $n_1 < n_2 < n_3 \neq N/2$ , then  $C(n_1, n_2, n_3)$  is the disjoint union of tori.*

*Proof.* Theorem 21 implies that our space satisfies the condition that each edge is contained in exactly two 2-simplices. The cyclic ordering of the 2-simplices containing a vertex  $\{a\}$  is evident in Figure 9, once we observe that the condition  $n_1 < n_2 < n_3$  implies that all six of the 2-simplices containing vertex  $a$  are distinct. Thus,  $C(n_1, n_2, n_3)$  is a surface.

We now orient all the 2-simplices in a consistent fashion, by taking the orientation of the edges to be by length, from least to greatest. This is a consistent orientation by Lemma 22, as one can easily see in Figure 10. The figure shows only the comparison between simplices which share an edge of length  $n_1$ , but the other two cases are similar. Thus,  $C(n_1, n_2, n_3)$  is an orientable surface.

Now recall that the Euler characteristic of  $C(n_1, n_2, n_3)$  is 0 by Theorem 12, and, if  $d = \gcd(n_1, n_2, n_3)$  then by Theorem 16, this Euler characteristic is  $d$  times the Euler characteristic of each of the components, so that each component of  $C(n_1, n_2, n_3)$  must also have Euler characteristic 0. By the classification of compact, connected, orientable surfaces (described for example in [3]), we see that any compact, connected, orientable surface with Euler characteristic 0 is a torus. Hence,  $C(n_1, n_2, n_3)$  is the disjoint union of tori.  $\square$

When the edge lengths of the 2-simplices are not all distinct, we can still have a surface.

**Theorem 24.** *If  $n_1 = n_2$  and  $n_3 = N/2$ , then  $n_3 = 2n_1$  and  $C(n_1, n_1, 2n_1)$  is the disjoint union of  $n_1$  spheres.*

*Proof.* The fact that  $n_3 = 2n_1$  follows directly from Lemma 5. It follows from Theorem 16 that we only need to compute  $C(1, 1, 2)$ . In  $C(1, 1, 2)$ , we have 4 faces:  $\{0, 1, 2\}$ ,  $\{0, 1, 3\}$ ,  $\{0, 2, 3\}$ , and  $\{1, 2, 3\}$ . Gluing these faces along common edges, we see the space  $C(1, 1, 2)$  in Figure 11.

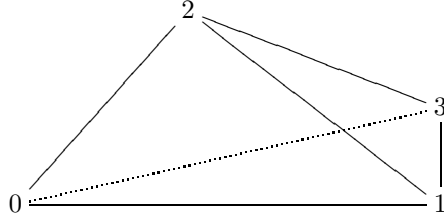


FIGURE 11. A picture of  $C(1, 1, 2)$ , a tetrahedron boundary.

These four 2-simplices are the boundary of the tetrahedron with vertices  $\{0, 1, 2, 3\}$ , and the boundary of a tetrahedron is topologically equivalent to a (2-dimensional) sphere. We note the nice ordering around all the vertices, just as described above.  $\square$

**Theorem 25.** *If  $C(n_1, n_2, n_3)$  is a surface without boundary, then it is either a disjoint union of tori or a disjoint union of spheres.*

*Proof.* Suppose  $C(n_1, n_2, n_3)$  is a surface without boundary. Since  $C(n_1, n_2, n_3)$  is a surface without boundary, each edge is contained in exactly two 2-simplices. Hence, by Theorem 21, each edge length  $n_i$  satisfies exactly one of the following two Conditions.

- (1)  $n_i = N/2$  and  $n_j = n_k$
- (2)  $n_i \neq N/2$  and  $n_j \neq n_k$

By Lemma 4,  $n_1 < N/2$  and  $n_2 < N/2$ , so Condition (2) applies to  $n_1$  and  $n_2$ , and we have  $n_1 \neq n_3$  and  $n_2 \neq n_3$ . This, combined with our standing assumption  $1 \leq n_1 \leq n_2 \leq n_3 < N$ , implies that  $1 \leq n_1 \leq n_2 < n_3 < N$ .

The integer  $n_3$ , on the other hand, is either equal to  $N/2$  or not equal to  $N/2$ . If  $n_3 = N/2$ , then we are in Condition (1), and  $n_1 = n_2$ , so Theorem 24 applies and  $C(n_1, n_2, n_3)$  is a disjoint union of spheres. If  $n_3 \neq N/2$ , then we are in Condition (2), and  $n_1 \neq n_2$ , so we have  $1 \leq n_1 < n_2 < n_3 \neq N/2$ , precisely the situation of Theorem 23, and  $C(n_1, n_2, n_3)$  is a disjoint union of tori.  $\square$

**6.2. Surfaces with Boundary.** The next simplest kind of space is a surface with boundary. In order for  $C(n_1, n_2, n_3)$  to have a boundary, there must be an edge contained in exactly one 2-simplex. In this section, we classify all such spaces whose connected components are either a 2-simplex, cylinder, or Möbius band. As before, we consider a general vertex  $a$  and compute the 2-simplices which contain it. We attach the 2-simplices along equal edges and determine the resulting spaces.

**Theorem 26.** *There are only 3 cases in which  $C(n_1, n_2, n_3)$  is a surface with boundary:*

- (1)  $C(n, n, n)$ .
- (2)  $C(n, n, n+k)$ , with  $n+k \neq N/2$ .
- (3)  $C(n, n+k, n+k)$ .

*Proof.* Suppose  $C(n_1, n_2, n_3)$  is a surface without boundary. We must have at least one edge which is contained in only one 2-simplex, and we must not have any edges contained in more than two 2-simplices. Theorem 21 implies that we must have  $n_i \neq N/2$  and  $n_j = n_k$  for some arrangement of the indices 1, 2, and 3 as  $i, j$  and  $k$ . This implies that either  $n_1 = n_3$ ,  $n_1 = n_2$  or  $n_2 = n_3$ . If  $n_1 = n_3$  we are in case (1) since  $n_1 \leq n_2 \leq n_3$  forces all three to be equal. Since  $N = 3n$ , the condition that  $n_2 \neq N/2$  is then automatic. If  $n_1 = n_2 \neq n_3$  then we are in case (2), with the condition  $n_3 \neq N/2$ . (We will find it convenient to write  $n_1 = n_2 = n$  and  $n_3 = n+k$  with  $k > 0$  in this case.) If  $n_1 \neq n_2 = n_3$ , then we have case (3), with the condition  $n_1 \neq N/2$  following from Lemma 4. Again, we find it convenient to write  $n_1 = n$  and  $n_2 = n_3 = n+k$ .

So far, we have proved that if  $C(n_1, n_2, n_3)$  is a surface with boundary, then it must be of the form (1), (2), or (3). In Theorems 27 and 28, we prove that each of these is indeed a surface with boundary.  $\square$

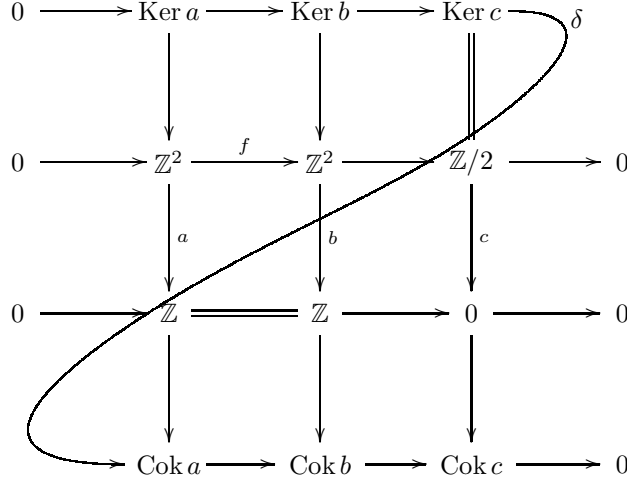


FIGURE 12. The Snake Lemma used in Lemma 30

In all three of these cases, it easily follows from Lemma 8 that there is no distinction between Type I and Type II 2-simplices.

**Theorem 27.** *The space  $C(n, n, n)$  is the disjoint union of  $n$  copies of a 2-simplex.*

*Proof.* By Theorem 16,  $C(n, n, n)$  is simply the disjoint union of  $n$  copies of  $C(1, 1, 1)$ . But  $C(1, 1, 1)$  is the 2-simplex with vertices  $0, 1, 2 \in \mathbb{Z}/3$ .  $\square$

In the remaining two cases from Theorem 26, the parity of  $N$  completely determines the space. We may assume that the  $\gcd(n, k) = 1$ , otherwise by Theorem 16, we'd be looking at a disjoint union of component spaces.

**Theorem 28.** *Assume that the  $\gcd(n, k) = 1$  and  $n+k \neq N/2$ . The spaces  $C(n, n, n+k)$  and  $C(n, n+k, n+k)$  are cylinders if  $N$  is even and Möbius bands if  $N$  is odd.*

Before we begin the proof, we use the Snake Lemma from homological algebra to prove a number theoretic fact which we need.

**Lemma 29** ((Snake Lemma)). *Let*

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c \\ 0 & \longrightarrow & D & \xrightarrow{f'} & E & \xrightarrow{g'} & F \longrightarrow 0 \end{array}$$

*be a commutative diagram of modules with exact rows. Then there exists an exact sequence relating the kernels and cokernels of  $a$ ,  $b$ , and  $c$ :*

$$(1) \quad 0 \longrightarrow \text{Ker } a \longrightarrow \text{Ker } b \longrightarrow \text{Ker } c \xrightarrow{\delta} \text{Cok } a \longrightarrow \text{Cok } b \longrightarrow \text{Cok } c \longrightarrow 0.$$

*Proof.* See [6].  $\square$

The situation we need, shown in Figure 12, has

$$f = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} n & k \end{bmatrix}, \quad \text{and so} \quad a = \begin{bmatrix} 3n+k & n+k \end{bmatrix}.$$

Therefore,  $\text{Cok } a = \mathbb{Z}/\gcd(3n+k, n+k)$  and  $\text{Cok } b = \mathbb{Z}/\gcd(n, k)$ , by Proposition 13. The  $\mathbb{Z}/2$  in Figure 12 occurs because the order of the cokernel of a homomorphism between free abelian groups is the determinant of the matrix representing the homomorphism (when that determinant is nonzero). Here, the determinant of  $f$  is  $\det(f) = 3 \cdot 1 - 1 \cdot 1 = 2$ .



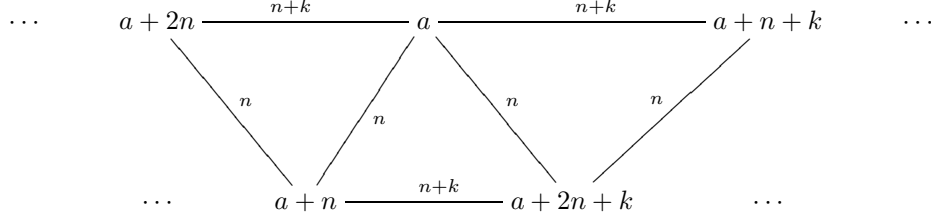


FIGURE 13. 2-simplices placed around the vertex  $a$  in  $C(n, n, n+k)$ .

The Snake Lemma exact sequence (1), is then the following long exact sequence.

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/2 \xrightarrow{\delta} \mathbb{Z}/\gcd(3n+k, n+k) \longrightarrow \mathbb{Z}/\gcd(n, k) \longrightarrow 0$$

**Lemma 30.** *Let  $n_0 = n/\gcd(n, k)$  and  $k_0 = k/\gcd(n, k)$ . Then*

$$\gcd(3n+k, n+k) = \begin{cases} 2\gcd(n, k) & \text{if } n_0 \text{ and } k_0 \text{ both odd} \\ \gcd(n, k) & \text{otherwise} \end{cases}$$

*Proof.* The image of  $\delta$  either has order 1 or order 2, and correspondingly,  $\gcd(3n+k, n+k)$  is either 1 or 2 times  $\gcd(n, k)$ . By exactness of the sequence, the ratio is 2 if and only if the map  $\mathbb{Z} \longrightarrow \mathbb{Z}$  of kernels is onto. Since the kernel of  $b$  is generated by  $[k_0 \ -n_0]^T$ , this is equivalent to asking whether or not we can solve

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} k_0 \\ -n_0 \end{bmatrix}$$

for integers  $x$  and  $y$ . Row reducing the matrix for  $f$  yields the following augmented matrix:

$$\left[ \begin{array}{cc|c} 1 & 1 & -n_0 \\ 0 & -2 & k_0 + 3n_0 \end{array} \right].$$

This is solvable in integers, and hence the ratio is 2, if and only if  $k_0 + 3n_0$  is even. Therefore, the  $\gcd(3n+k, n+k) = 2\gcd(n, k)$  if and only if  $3n_0 + k_0$  is even, and  $\gcd(3n+k, n+k) = \gcd(n, k)$  if and only if  $3n_0 + k_0$  is odd.  $\square$

We are now prepared to finish the proof of Theorem 28, completing the proof of Theorem 26.

*Proof.* By Theorem 21 and the proof of Theorem 26, each edge is contained in either one or two 2-simplices, so that the condition on edges for a surface with boundary is satisfied. Next, we will show that the 2-simplices containing each vertex have a linear order, verifying the other condition for a surface with boundary, and showing that all the vertices are on the boundary. In the process we will observe that the 2-simplices are arranged as in Mazzola's harmonic strip, and hence form either a Möbius band or cylinder.

We consider  $C(n, n, n+k)$  first. Eliminating redundancies from the list of 2-simplices following the proof of Theorem 21, we find that the following 2-simplices contain the vertex  $a$ .

- (1)  $\{a, a+n, a+2n\}$
- (2)  $\{a, a+n, a+2n+k\}$
- (3)  $\{a, a+n+k, a+2n+k\}$

(Recall that  $n_1 = n_2 = n$  and  $n_3 = n+k$  here.)

We can arrange these around the vertex  $a$  in a consistent, linear fashion, as shown in Figure 13. So we now know that  $C(n, n, n+k)$  is a surface without boundary. We also note the side lengths, as this will be useful in our analysis.

Note that the boundary edges are those of length  $n+k$ , since each edge of length  $n_3 = n+k \neq N/2$  lies in only one 2-simplex, as shown in Theorem 21.

The space  $C(n, n, n+k)$  can be formed by starting with a 2-simplex  $\{a, a+n, a+2n+k\}$  and moving right in Figure 13 until this 2-simplex reappears. (Recall that  $\gcd(n, n+k) = 1$  implies that  $C(n, n, n+k)$  is connected by Corollary 15, so we know this will occur.)

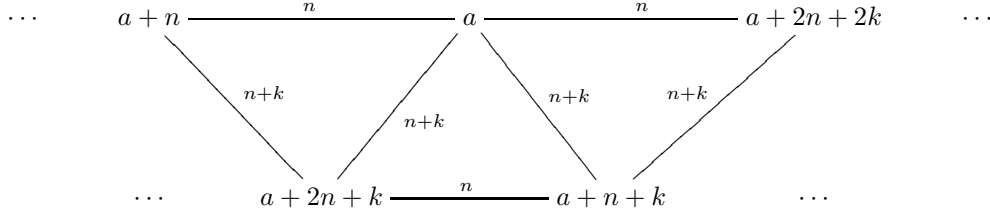


FIGURE 14. 2-simplices placed around the vertex  $a$  in  $C(n, n+k, n+k)$ .

The only question remaining is whether the edges are joined preserving or reversing orientation, when we return to the starting 2-simplex. This is the same as asking whether the boundary has one or two components, since we already know that  $C(n, n, n+k)$  is a surface without boundary by [3].

Two vertices lie in the same component of the boundary if they can be connected by repeatedly adding  $n+k$  to one to get the other. Suppose  $N$  is even. Since  $\gcd(N, n+k) = \gcd(3n+k, n+k) = 2$ , there are two cosets for the subgroup generated by  $n+k$  in  $\mathbb{Z}/N$ , and hence the boundary has two components. This implies that the space is a cylinder. By Lemma 30, this occurs when  $n$  and  $k$  are both odd, so that  $N = 3n+k$  is even.

Suppose  $N$  is odd. Since  $n$  and  $k$  cannot both be even, by our assumption that  $\gcd(n_1, n_2, n_3) = \gcd(n, k) = 1$ , the only possibility is that exactly one of  $n$  and  $k$  is odd, since  $N$  is odd. In this case the subgroup of  $\mathbb{Z}/N$  generated by  $n+k$  is the whole group, so the boundary has only one component, and the space is a Möbius band.

For  $C(n, n+k, n+k)$ , we again consider the list of 2-simplices which contain the vertex  $a$ , eliminating the redundant cases as before.

- (1)  $\{a, a+n, a+2n+k\}$
- (2)  $\{a, a+n+k, a+2n+2k\}$
- (3)  $\{a, a+n+k, a+2n+k\}$

Once again, we arrange these around the vertex  $a$ , noting that each edge of length  $n$  is contained in exactly one 2-simplex by Theorem 21, as shown in Figure 14.

Two vertices lie in the same component of the boundary if they can be connected by repeatedly adding  $n$  to one to get the other. Hence, we now consider  $\gcd(N, n) = \gcd(3n+2k, n)$  to see how many components the boundary has.

Using the fact that  $\gcd(3b+a, b) = \gcd(a, b)$ , we see that  $\gcd(3n+2k, n) = \gcd(2k, n)$ . Since  $\gcd(n_1, n_2, n_3) = \gcd(n, k) = 1$ ,

$$\gcd(N, n) = \gcd(3n+2k, n) = \gcd(2k, n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

Since  $N$  and  $n$  have the same parity, we have one boundary component, and hence a Möbius band, when  $N$  is odd, and two boundary components, and hence a cylinder, when  $N$  is even.  $\square$

**6.3. Not a surface.** Using the conditions on the  $n_i$  of Theorem 21, Theorems 25 and 26 imply we have now classified all of the cases when  $C(n_1, n_2, n_3)$  will be a surface or a surface with boundary. In the remaining cases,  $n_3 = N/2$  and  $n_1 \neq n_2$  by Theorem 21. By Lemma 5, these are the spaces  $C(n_1, n_2, n_1+n_2)$ , with  $n_1 < n_2$ . By Theorem 16 we may assume that  $\gcd(n_1, n_2) = 1$ .

**Theorem 31.** *The simplicial complex  $C(n_1, n_2, n_1+n_2)$  with  $n_1 < n_2$  is a circle of  $n_1+n_2$  tetrahedra boundaries, each joined to the ones on either side of it along opposite edges.*

*Proof.* In  $C(n_1, n_2, n_1+n_2)$ , we see that each vertex  $a$  lies in a unique interval of length  $N/2$   $\{a, a+n_1+n_2\}$ . By Theorem 21, this edge is contained in exactly four 2-simplices, two of which are in each of the following tetrahedra boundaries.

- (1)  $\{a, a+n_1, a+n_1+n_2, a+2n_1+n_2\}$
- (2)  $\{a, a+n_2, a+n_1+n_2, a+n_1+2n_2\}$

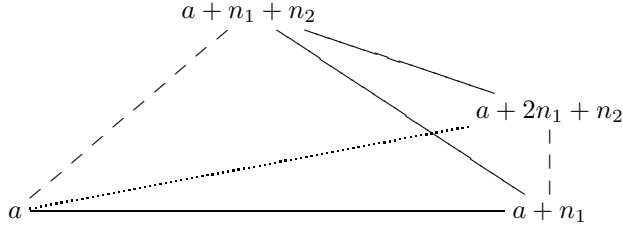


FIGURE 15. One tetrahedron boundary in the circle of tetrahedra boundaries. The tritones are indicated by dashed lines.

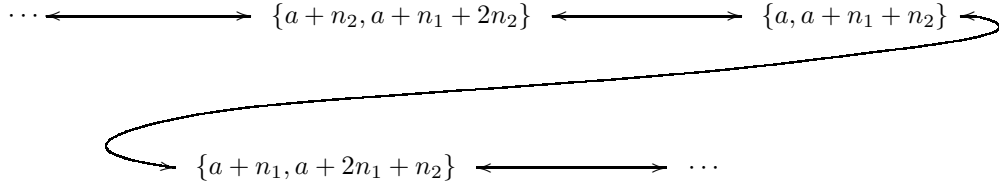


FIGURE 16. Relation between tritones in a circle of tetrahedra boundaries: to move right, add  $n_1$  to each vertex in the tritone, and to move left, add  $n_2$  to each vertex in the tritone.

All four of the 2-simplices in each of these tetrahedra boundaries are in  $C(n_1, n_2, n_1 + n_2)$ . The intersection of these two tetrahedra boundaries is exactly the edge corresponding to the tritone  $\{a, a + n_1 + n_2\}$ . The tetrahedra boundaries can thus be envisioned as linking tritones in a chain in which we add  $n_1$  in one direction and add  $n_2$  in the reverse direction (see Figure 16).

This chain forms a circle because each tritone lies in only two tetrahedra boundaries, and since  $\gcd(n_1, n_2, n_1 + n_2) = 1$ , the space must be connected. (Alternatively, there are  $N/2$  tritones which we can index by the integers modulo  $N/2 = n_1 + n_2$ . Since  $n_1$  is relatively prime to  $N/2$ , repeatedly adding  $n_1$  must exhaust the tritones.)

□

As noted earlier, the circle of tetrahedra boundaries is a quotient of Peck's Klein bottle [16]. In fact, the Klein bottle is a desingularization of the circle of tetrahedra boundaries. Precisely, one obtains the circle of tetrahedra boundaries by identifying the two distinct copies of each tritone which occur in Peck's Klein bottles. The 'twist' in the space vanishes when this is done.

## 7. SUMMARY AND CONCLUSION

We have completely classified  $C(n_1, n_2, n_3)$  in any case.

**Theorem 32.** *The connected components of the generalized Tonnetz  $C(n_1, n_2, n_3)$  are isomorphic and are 2-simplices, tetrahedra boundaries, tori, cylinders, Möbius bands, or circles of tetrahedra boundaries. See Figure 1 for the precise cases.*

The generic case is  $n_1 < n_2 < n_3$  with  $n_3 \neq N/2$ , and this is shown in Theorem 23 to give a disjoint union of tori. When  $n_3 = N/2$ , we must have  $n_3 = n_1 + n_2$  by Lemma 5. In the most generic of these cases,  $n_1 \neq n_2$ , the torus is 'pinched' along edges corresponding to tritones to form a 'circle of tetrahedra boundaries' (Theorem 31). These are exactly the cases considered by Peck [16] in his Klein bottle *Tonnetz*, which are desingularizations of ours.

In the remaining cases, either  $n_1 = n_2 < n_3$ ,  $n_1 < n_2 = n_3$ , or  $n_1 = n_2 = n_3$ . When  $n_1 = n_2$ , we have the possibility that  $n_3 = N/2$ , in which case we have a disjoint union of individual tetrahedra boundaries (i.e., 2-spheres) (Theorem 24). (This can be considered a limiting case of the circle of tetrahedra boundaries, in which the circle has only one tetrahedron boundary.) More generically,  $n_3 \neq N/2$  and we obtain a generalization of Mazzola's harmonic strip, which we identify as either a Möbius band or cylinder, according

to the parity of the number of notes in the octave (Theorem 28). When  $n_1 < n_2 = n_3$ , the possibility that  $n_3 = N/2$  no longer exists and we again have harmonic strips (Theorem 28). Finally, if  $n_1 = n_2 = n_3$ , we are discussing augmented triads  $\{C, E, G^\#\}$  and their microtonal transpositions (Theorem 27).

By re-envisioning the *Tonnetz* as a simplicial complex, rather than as a quotient of a planar region, we have sharpened the focus on the relationships between the chords in the *Tonnetz* by using exactly (and only) the relations between the intervals in the chords to assemble the space. This abstraction allows us to define generalized *Tonnetze* for triads of arbitrary shape in scales which divide the octave into any number of steps.

When the chord has a ‘common denominator’, i.e., in the cases  $C(dn_1, dn_2, dn_3)$ , the resulting *Tonnetz* is a disjoint union of identical subspaces which are generalized *Tonnetze*  $C(n_1, n_2, n_3)$  in their own right. The  $T/I$  group then cycles through these components, while the  $P$ ,  $L$ , and  $R$  operations are confined to each component.

Interestingly, all possibilities except the Möbius band occur in the chromatic scale  $\mathbb{Z}/12$ , as noted in the Introduction. (The Möbius band can only occur in scales which divide the octave into an odd number of steps.)

#### ACKNOWLEDGEMENT(S)

I would like to thank Sean Gavin for all his helpful ideas and advice, as well as Tom Fiore for his discussions and expertise in both mathematics and music theory. I want to thank Robert Bruner for all his guidance and help with this paper. Without his time and patience, this work would not have been possible. I would also like to thank a very diligent referee, whose numerous and detailed suggestions greatly improved the content and readability of this paper. This work was funded in part by the Office of Undergraduate Research of Wayne State University. It was also funded by NSF grant PHY-0851678.

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